

# Analysis and Mitigation of Pose Estimation Uncertainty on SE(3) for Magnetic Localization

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## I. INTRODUCTION

**Abstract**—Magnetic localization, owing to its immunity to line-of-sight occlusion and noncontact nature, is considered a promising technology for medical applications. While it is intuitive that localization performance degrades as the target moves farther from the sensor array, uncertainty analysis has long been overlooked, which is essential for quantifying localization quality. In this work, we present a pose estimation and uncertainty analysis framework on SE(3) for magnetic source localization using sensor arrays, which enables concise formulation of the problem and quantitative assessment of the results. The volume of the uncertainty ellipsoid is used to characterize localization confidence, while the surface shape in Cartesian space is used for visualization. This also provides insight into the effective workspace of the magnetic localization system prior to deployment. Guided by this analysis, we designed a movable magnetic sensor array to expand the limited sensing volume and mitigate localization uncertainty, thereby enhancing overall localization performance. Simulations and pose tracking experiments validate the effectiveness of this framework. By dynamically moving the sensor array to minimize the volume of the uncertainty ellipsoid, localization errors are reduced compared with static and projection-based strategies by 67.04%, 16.51% in position and 43.87%, 20.73% in orientation, respectively. Furthermore, phantom experiments on distal locking screw alignment and magnetic capsule endoscope tracking demonstrate the system's capability in improving localization accuracy (reducing alignment errors by 71.88%) and expanding the effective workspace by several folds.

**Index Terms**—Magnetic localization, matrix lie groups, medical robots and systems, uncertainty analysis.

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POSE estimation has long been a fundamental topic of discussion and research, playing a critical role in navigation, manipulation, human-machine interaction, and biomedical engineering. Conventional localization techniques include optical tracking [1], LiDAR [2], ultrasound [3], and radio frequency-based methods [4]. Each of these approaches has its own merits and limitations, but they commonly face challenges, such as occlusion and signal interference in complex environments.

Magnetic-based localization techniques have attracted increasing attention due to their robustness against visual occlusion and most material blockages, as well as their high accuracy and real-time capability. The basic principle of magnetic localization is to estimate the relative pose of a target by solving the inverse problem based on the sensor measurements and a forward model of the magnetic field. In such systems, the sensors—typically magnetometers or miniature induction coils—detect magnetic fields generated by either permanent magnets or electromagnetic coils. With the rise of *in vivo* miniature medical robots and their expanding clinical promise [5], [6], [7], [8], [9], magnetic localization is emerging as a radiation-free modality capable of delivering real-time intraoperative state feedback. This article will mainly focus on a class of magnetic localization systems that estimate the pose of a permanent magnet using an array of magnetic sensors. Since the target magnet does not require wired connections or power consumption, these kinds of systems offer high flexibility and strong environmental adaptability.

Extensive research has been conducted on magnetic localization systems based on sensor arrays for tracking permanent magnets. One prominent application is in capsule endoscopy robots [10], where the embedded magnet in the capsule is localized using external magnetic sensor array. Some studies further exploit the fact that the localized magnet can also interact with external magnetic fields, enabling simultaneous wireless localization and actuation [11]. In other cases, magnetic localization has been employed as a tool for characterizing muscle physiological activity [12] or even as a control signal, allowing amputees to precisely operate prosthetic limbs [13].

However, most of these studies primarily emphasize absolute localization accuracy, with limited attention paid to the reliability or confidence of the localization results. While it is intuitive that localization errors tend to increase with distance from the sensor array, distance alone does not fully determine localization

quality. Other influential factors include the spatial configuration of the sensor array, the magnet's magnetization characteristics, and its orientation relative to the sensors.

In practical applications, it is often difficult to assess the credibility of localization results in the absence of ground truth data. Therefore, it is essential to establish a quantitative method for evaluating the workspace and reliability of magnetic localization. Such a method would provide a standardized framework for researchers to assess the effective operational space and localization performance of magnetic systems in a consistent and generalizable manner.

Therefore, the discussion of pose uncertainty in magnetic localization is of critical importance. Slawinski et al. specifically investigated how localization uncertainty impacts the control of magnetic robots and introduced methods to reduce force control uncertainty [14]. While prior work on uncertainty in magnetic localization is limited, several notable studies have been conducted. For instance, Edridge et al. employed a magnetometer array to implement odometry and derived an analytical expression for the covariance of pose variations [15]. Daroogheha et al. used an unscented Kalman filter in a system, where multiple magnetic dipoles are utilized to localize sensor positions, yielding covariance ellipses for the estimated locations, which are applied to both robotic arm and mobile robot tracking [16]. Shen et al. developed an active electromagnetic tracking system, and through Jacobian analysis, identified that localization sensitivity is maximized along the electromagnetic coil's axis [17]. Although these studies incorporated covariance estimation or sensitivity analysis in the context of position and orientation tracking, there remains a significant gap in the literature regarding uncertainty analysis for the localization of a permanent magnet using a magnetic sensor array. Moreover, there is still a lack of systematic, quantitative evaluation of the operational workspace and localization reliability for such systems. This gap highlights the need for a unified method to assess uncertainty and performance in magnetic localization of permanent magnets.

In addition, most current magnetic localization systems employ fixed sensor arrays, with research efforts largely focused on localization and optimization within the effective workspace of these static configurations. Consequently, to compensate for poor localization quality at the periphery of the array, some works resort to designing large-scale sensor arrays to achieve wider coverage [10], [11], [18]. Although expanding a sensor array to encompass the entire workspace is a viable strategy for increasing detection volume, it entails fundamental limitations. Due to the rapid inverse-cube attenuation of magnetic field strength ( $B \propto r^{-3}$ ), the majority of sensors in a fully covered array would be situated too far from the source to provide effective measurements. Although Xu et al. employed a similar targeted strategy by adaptively activating a sensor subset [19], the inherent discrete spacing (60 mm) limits the system's capacity to achieve theoretically optimal tracking. More importantly, this results in substantial resource wastage, as only a small subset contributes to pose estimation at any given instant. This inefficiency is particularly pronounced when tracking small-scale targets with low magnetic moments. Furthermore, large

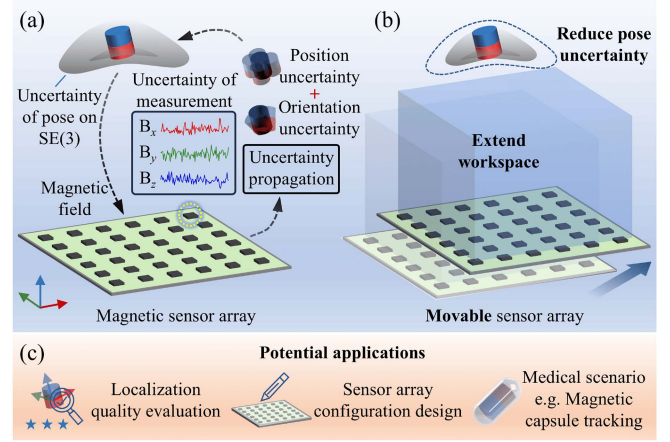


Fig. 1. Schematic of the proposed magnetic localization uncertainty analysis and reduction method. (a) The uncertainty propagates from sensor noise to the pose of the target. (b) By moving the sensor array, the effective workspace is expanded, and the uncertainty in localization is reduced. (c) The proposed analytical method and system have potential applications, including real-time assessment of localization quality, guiding the design of the array, and in medical scenarios such as tracking magnetic capsules.

monolithic arrays are cumbersome to manufacture and deploy [19], while modular alternatives inevitably introduce systematic errors through intermodule alignment uncertainties [11].

A promising alternative is the use of dynamic sensor arrays, which allow for flexible repositioning of a compact printed circuit board (PCB) to selectively target regions of interest. This approach can effectively extend the operational workspace, while maintaining high localization accuracy with fewer sensors. A similar concept has also been validated in kilometer-scale environments as an efficient method that achieves high-resolution localization over large areas with only limited hardware deployment [20], [21].

To address the aforementioned research gaps, this work presents a detailed uncertainty analysis of localizing a permanent magnet using a magnetic sensor array. This enables a quantitative evaluation of localization quality across different regions of the workspace. Furthermore, With the goal of reducing localization uncertainty, we utilize the planar mobility for the sensor array, dynamically repositioning to enlarge the effective workspace and mitigate localization uncertainty, particularly in peripheral regions where performance typically degraded, as shown in Fig. 1.

Central to our methodology is the systematic treatment of pose representation and uncertainty propagation within the special Euclidean group SE(3). It is worth noting that, while classical representations like Euler angles and quaternions are common in magnetic tracking [11], [18], they impose some limitations. Euler angles suffer from gimbal lock singularities and complex trigonometric derivatives [22], whereas quaternions introduce an overparameterized state space (requiring unit-norm constraints) that complicates covariance propagation [23]. In contrast, we leverage SE(3) and its Lie algebra  $\mathfrak{se}(3)$  for state representation, pose estimation, and uncertainty analysis. This approach not only enables concise and rigorous expression of derivation, but also allows uncertainty to be defined linearly on the tangent

space of the motion manifold, ensuring numerical stability and analytical clarity. We can therefore achieve a compact representation of the pose estimation problem, including its gradients for accelerating computation, and map the uncertainty from the Lie algebra space to Cartesian coordinates for intuitive visualization.

The main contributions of this article are summarized as follows.

- 1) To the best of our knowledge, the representation, estimation, and uncertainty analysis framework for magnetic localization on SE(3) is proposed for the first time. Localization uncertainty is formulated by propagating sensor noise into the pose perturbation.
- 2) A planar mobile magnetic sensor array is designed to dynamically mitigate localization uncertainty, thereby enabling workspace expansion and improving accuracy.
- 3) The proposed framework and system are validated through trajectory tracking experiments and localization tasks in clinical scenarios including interlocking nail surgery and gastrointestinal capsule endoscopy.

The rest of this article is organized as follows: Section II formulates the problem and derives the pose estimation and propagation of uncertainty on SE(3). In Section III, we conduct simulation analysis within the workspace under different magnet poses and sensor array configurations, and demonstrate the benefits of planar mobility. Section IV presents the results of pose tracking and comparative experiments, as well as demonstrations of magnetic localization-guided alignment of locking screws in interlocking nail surgery and tracking of a magnetic capsule. Finally, Section V concludes this article.

## II. METHODOLOGY

### A. Problem Formulation

In this study, we aim to derive the uncertainty of the pose estimation and propose a method to mitigate this uncertainty by adjusting the position of the movable sensor array. We assume that the uncertainty in pose estimation primarily originates from Gaussian noise  $\mathbf{n}_s^i \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_s^i)$  distributed along each of the three axes of the magnetometers in the sensor array. Furthermore, we also model the localization uncertainty, expressed in the Lie algebra space, as a zero-mean Gaussian distribution,  $\mathbf{n}_m \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_m)$ , which is a key focus in our analysis.

As illustrated in Fig. 2, four coordinate frames are involved in the system: the world frame  $\{w\}$ , the PCB frame  $\{b\}$ , the magnet frame  $\{m\}$ , and the local frame of the  $i$ -th sensor  $\{s^i\}$ , where  $i = 1, \dots, N$ . We denote the representation of a variables  $\mathbf{Y}$  in frame  $x$  as  ${}^x\mathbf{Y}$ . It is important to note that since a rigid and grid-structured sensor array is used, the relative positions of the sensors are determined by the initial PCB layout and remain constant throughout operation. Therefore, the pose of PCB frame uniquely determines the poses of all individual sensors in the array.

The cylindrical permanent magnet generates a magnetic field that can be well described using a mathematical model, as shown in Fig. 2. This field is measured by the triaxial magnetometers in the sensor array. Given these measurements, the magnet's pose is estimated by solving an inverse problem using a gradient-based

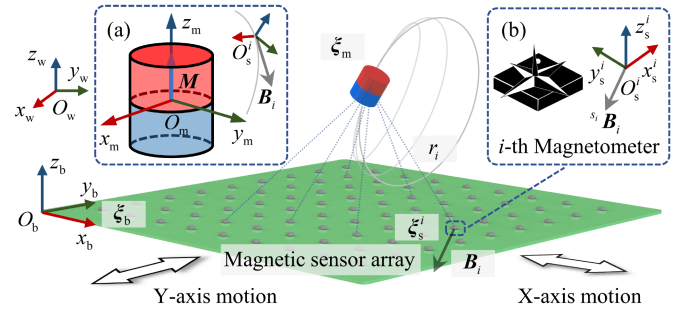


Fig. 2. Illustration of the proposed magnetic localization system with a movable sensor array and the coordinate frames involved in the analysis. (a) Coordinate system of the magnet and its generated magnetic field. (b) Local coordinate system of the tri-axis magnetometer.

optimization algorithm. It is worth noting that the proposed pose estimation and uncertainty analysis method is not limited to cylindrical magnets. We adopt them as they are the most commonly used geometry in related studies.

Since we adopt Lie algebra for pose representation, which is not closed under addition. Thus, we need to employ the exponential map  $\text{Exp}(\cdot)$  to transform from  $\mathfrak{se}(3)$  to SE(3), enabling pose composition through matrix multiplication. This motivates our representation of pose uncertainty also in the Lie algebra space, where perturbations are applied via right multiplication to the pose mean. By treating the pose of the magnet as a random variable, it can be expressed as

$${}^w_m\mathbf{T} = {}^w_m\bar{\mathbf{T}} \cdot \text{Exp}(\mathbf{n}_m) \quad (1)$$

where  ${}^w_m\mathbf{T} = \text{Exp}(\xi_m)$ ,  $\xi_m \in \mathfrak{se}(3)$ , and  $\bar{(\cdot)}$  represent the mean of a random variable.

In the subsequent analysis, we first introduce an analytical magnetic field model for a cylindrical magnet based on complete elliptic integrals, along with its spatial gradient. Using this model, we derive a measurement function and its Jacobian with respect to the magnet's pose. This formulation enables efficient pose estimation using nonlinear optimization. We then propagate the sensor noise to the uncertainty of the pose perturbation by linearizing a reformulated measurement function. The covariance matrix of the resulting perturbation defines an uncertainty ellipsoid, whose volume is used as the objective function to optimize the sensor array's planar position.

### B. Magnetic Field Distribution and Gradient

The localization system relies on the magnetic field generated by a permanent magnet. To achieve a more accurate tracking performance, we adopt a generalized complete elliptic integral model [24] rather than the commonly used magnetic dipole model, which is inaccurate near the source [25].

Consider a cylindrically shaped permanent magnet with axial magnetization  $M$ , characterized by a half-length  $L$  and radius  $R$ . The magnetic flux density  $\mathbf{B}$ , expressed in cylindrical coordinates, is given in closed form as follows:

$$B_\rho = \frac{\mu_0 M R}{\pi} [\alpha_+ P_1(k_+) - \alpha_- P_1(k_-)] \quad (2a)$$



$$B_z = \frac{\mu_0 MR}{\pi(\rho + R)} [\beta_+ P_2(k_+) - \beta_- P_2(k_-)] \quad (2b)$$

where  $P_1$  and  $P_2$  are the auxiliary functions, given in the Appendix A.

For clarity in the subsequent discussion, the magnetic field on the operating point  $\mathbf{r}$  is represented by  $\mathcal{B}(\mathbf{r})$ , and the magnetic field representation is transformed from cylindrical to Cartesian coordinates to facilitate subsequent computations

$$B_x = x \frac{B_\rho}{\rho} \quad (3a)$$

$$B_y = y \frac{B_\rho}{\rho}. \quad (3b)$$

The linearization of the measurement model requires the computation of the Jacobian matrix, which involves the magnetic field gradient matrix  $\mathcal{H}$ . At an operating point  $\mathbf{r}$  in the magnet's local frame, the magnetic gradient is given by

$$\mathcal{H}(\mathbf{r}) = \begin{bmatrix} \partial_x B_x & \partial_y B_x & \partial_z B_x \\ \partial_x B_y & \partial_y B_y & \partial_z B_y \\ \partial_x B_z & \partial_y B_z & \partial_z B_z \end{bmatrix}. \quad (4)$$

Considering the four constraints imposed by Maxwell's equations under static magnetic fields, the magnetic gradient matrix becomes symmetric. Accordingly, five independent elements are selected for analysis:  $\partial_x B_x, \partial_y B_y, \partial_z B_z, \partial_x B_y$ , and  $\partial_x B_z$ . The detailed derivation of the partial derivative for the magnetic field can be found in Appendix A.

### C. Maximum-Likelihood Estimate of the Pose

We assume that the ambient magnetic field is known and can be subtracted through prior calibration. Thus, in the magnetic sensor array, the theoretical three-axis magnetic flux density measured by  $i$ th sensor in its own local coordinate frame is given by

$$\begin{aligned} \mathbf{B}_i &= \mathbf{s}_i^T \mathbf{R} \cdot \mathcal{B}(\mathbf{r}_i) \\ &= \mathbf{D} \cdot \text{Exp}(-\boldsymbol{\xi}_s^i) \text{Exp}(\boldsymbol{\xi}_m) \cdot \mathbf{D}^T \\ &\quad \cdot \mathcal{B}(\mathbf{D} \cdot \text{Exp}(-\boldsymbol{\xi}_m) \text{Exp}(\boldsymbol{\xi}_s^i) \cdot \mathbf{E}) \end{aligned} \quad (5)$$

where  $\mathbf{D} = [\mathbf{1}_{3 \times 3} \quad \mathbf{0}_{3 \times 1}]$  is a row-selection matrix used to extract the first three rows, and  $\mathbf{E} = [\mathbf{0}_{1 \times 3} \quad \mathbf{1}]^T$  is a column-selection matrix used to extract the last column.

Similar to general optimization problems, the difference between the theoretical and measured values from each triaxial magnetic sensor is concatenated into a single long column vector to form the objective function. Considering the sensor noise, in particular that the three axes of some magnetic sensors exhibit different noise levels, we assign different weights to the individual components of the residual vector. By exploiting the covariance matrix of the sensor noise, we obtain a maximum-likelihood estimate (MLE) under known noise statistics, with the pose  $\boldsymbol{\xi}_m$  of the magnet as the optimization variable

$$\min f(\boldsymbol{\xi}_m) = \frac{1}{2} (\tilde{\mathbf{B}} - \mathbf{B})^T \mathbf{R}_s^{-1} (\tilde{\mathbf{B}} - \mathbf{B}) \quad (6)$$

where the vector  $\mathbf{B} \in \mathbb{R}^{3N \times 1}$  is formed by vertically concatenating the magnetic field vectors  $\mathbf{B}_i \in \mathbb{R}^{3 \times 1}$  at each of the  $N$  sensors, the tilde symbol denotes the measured values, and  $\mathbf{R}_s = \text{Diag}(\mathbf{R}_s^1, \mathbf{R}_s^2, \dots, \mathbf{R}_s^N)$  is the measurement noise covariance matrix. Compared with ordinary least squares, the MLE is statistically more efficient, achieving the minimum variance among unbiased estimators.

Given the nonlinearity and complexity of this problem, the performance of the optimization algorithm is highly sensitive to the accuracy of the gradient information. By providing the analytical Jacobian matrix of the objective function, the gradient computation can be significantly improved, resulting in faster convergence and enhanced robustness. Leveraging the Lie algebra-based pose representation and the magnetic gradient matrix derived in the previous section, the Jacobian  $\mathbf{J}$  can be computed efficiently and concisely via the chain rule as follows:

$$\begin{aligned} \mathbf{J} &= \frac{\partial f(\boldsymbol{\xi}_m)}{\partial \boldsymbol{\xi}_m} = \mathbf{R}_s^{-1} (\mathbf{B} - \tilde{\mathbf{B}})^T \frac{\partial \mathbf{B}}{\partial \boldsymbol{\xi}_m} \\ &= \sum_{i=1}^N \left( \mathbf{R}_s^{i-1} (\mathbf{B}_i - \tilde{\mathbf{B}}_i)^T \frac{\partial \mathbf{B}_i}{\partial \boldsymbol{\xi}_m} \right). \end{aligned} \quad (7)$$

Here, we can take the partial derivative of (5) to obtain

$$\begin{aligned} \frac{\partial \mathbf{B}_i}{\partial \boldsymbol{\xi}_m} &= \mathbf{D} \cdot \text{Exp}(-\boldsymbol{\xi}_s^i) \left( \frac{\text{Exp}(\boldsymbol{\xi}_m) \cdot \mathbf{U}}{\partial \boldsymbol{\xi}_m} \right. \\ &\quad \left. + \text{Exp}(\boldsymbol{\xi}_m) \cdot \mathbf{D}^T \cdot \frac{\partial \mathcal{B}(\mathbf{D}\mathbf{V})}{\partial \mathbf{D}\mathbf{V}} \cdot \mathbf{D} \frac{\partial \mathbf{V}}{\partial \boldsymbol{\xi}_m} \right) \\ &= \mathbf{D} \cdot \text{Exp}(-\boldsymbol{\xi}_s^i) ((\text{Exp}(\boldsymbol{\xi}_m) \cdot \mathbf{U})^\odot \mathcal{J}(\boldsymbol{\xi}_m) \\ &\quad - \text{Exp}(\boldsymbol{\xi}_m) \cdot \mathbf{D}^T \cdot \mathcal{H}(\mathbf{D}\mathbf{V}) \cdot \mathbf{D} \cdot \mathbf{V}^\odot \mathcal{J}(-\boldsymbol{\xi}_m)) \end{aligned} \quad (8)$$

where we denote  $\mathbf{U} = \mathbf{D}^T \cdot \mathcal{B}(\mathbf{D}\mathbf{V})$  and  $\mathbf{V} = \text{Exp}(-\boldsymbol{\xi}_m) \text{Exp}(\boldsymbol{\xi}_s^i) \cdot \mathbf{E}$  for notational simplicity. The operator  $(\cdot)^\odot$  maps a homogeneous coordinate  $\mathbf{p} = \begin{bmatrix} \mathbf{a} \\ b \end{bmatrix} \in \mathbb{R}^4$ , where  $\mathbf{a} \in \mathbb{R}^3$  and  $b \in \mathbb{R}$ , to  $\mathbf{p}^\odot = \begin{bmatrix} b\mathbf{1}_{3 \times 3} & -\mathbf{a}^\wedge \\ \mathbf{0}_{1 \times 3} & \mathbf{0}_{1 \times 3} \end{bmatrix} \in \mathbb{R}^{4 \times 6}$ . The  $\mathcal{J}(\cdot)$  represents the left Jacobian of SE(3), enabling efficient handling of the nonlinear mapping between the Lie group and its associated Lie algebra. Its explicit form can be found in [26, Chap. 8].

Therefore, for the weighted nonlinear least-squares problem defined in (6), once the first-order derivative of the objective function is obtained as (7) and (8), numerical optimization methods such as the Gauss–Newton or Levenberg–Marquardt algorithm [27], which rely on gradient and quasi-second-order information, can be applied to ensure efficient convergence in the parameter space. Moreover, owing to the advanced pose representation on SE(3), the form of Jacobian matrix is significantly simplified compared with [18].

### D. Uncertainty Propagation

At this point, the pose of the magnetic target has been estimated through the optimization procedure. While most previous studies stop at this stage, in many practical applications, it is crucial to evaluate the reliability of the obtained result.



Intuitively, a magnetic sensor array exhibits varying localization confidence depending on the pose and characteristics of the target. Therefore, quantifying the localization uncertainty is essential for reliable deployment of magnetic localization systems, especially in real-time applications. A high level of uncertainty may indicate that the current system configuration is suboptimal, or that the target has moved beyond the effective workspace.

As defined in (1), the actual pose of the magnet is modeled as a random variable on the Lie group  $SE(3)$ , with uncertainty arising from zero-mean Gaussian noise  $\mathbf{n}_m$  represented in the associated Lie algebra. This noise acts directly on the pose mean, introducing both translational and rotational perturbations. The benefit of this formulation lies in the ability to represent pose uncertainty in a mathematically tractable manner, as defining covariance directly on  $SE(3)$  is generally intractable.

More importantly, magnetic localization frequently encounters reduced degree of freedom (DoF), where the magnet inherently has only 5 DoF. The rotation about its own longitudinal axis does not affect the surrounding magnetic field. Because the perturbation is explicitly separated, uncertainty in each DoF can be conveniently decoupled. Consequently, the missing dimension can be excluded from the uncertainty analysis by either omitting the corresponding row and column in the covariance matrix or directly setting them to zero. This approach allows us to represent the 5-DoF uncertainty using the first five dimensions of the perturbation, ensuring that the estimated uncertainty remains invariant to rotations about the symmetry axis, and thereby avoiding singularities caused by symmetry.

We reformulate the measurement model from (5) as a function of the pose with additive noise. Substituting the perturbed pose into the measurement model yields the forward relation between sensor measurements and noise

$$\begin{aligned} B_i &= {}^{s_i}R \cdot \mathcal{B}({}^m r_i) \\ &= D \cdot \text{Exp}(-\xi_s^i) \text{Exp}(\xi_m) \text{Exp}(\mathbf{n}_m) \cdot D^T \\ &\quad \cdot \mathcal{B}(D \cdot \text{Exp}(-\mathbf{n}_m) \text{Exp}(-\xi_m) \text{Exp}(\xi_s^i) \cdot E) \end{aligned} \quad (9)$$

By linearizing the above expression, we derive the Jacobian of the measurement with respect to the pose noise

$$\begin{aligned} J_n^i &= \frac{\partial B_i}{\partial \mathbf{n}_m} \\ &= D \cdot \text{Exp}(-\xi_s^i) \text{Exp}(\xi_m) \left( \frac{\text{Exp}(\mathbf{n}_m) \cdot U_n}{\partial \mathbf{n}_m} \right. \\ &\quad \left. + \text{Exp}(\mathbf{n}_m) \cdot D^T \cdot \frac{\partial \mathcal{B}(D \mathbf{V}_n)}{\partial D \mathbf{V}_n} \cdot D \frac{\partial \mathbf{V}_n}{\partial \mathbf{n}_m} \right) \\ &= D \cdot \text{Exp}(-\xi_s^i) \text{Exp}(\xi_m) ((\text{Exp}(\mathbf{n}_m) \cdot U_n)^\odot \mathcal{J}(\mathbf{n}_m) \\ &\quad - \text{Exp}(\mathbf{n}_m) \cdot D^T \cdot \mathcal{H}(D \mathbf{V}_n) \cdot D \cdot \mathbf{V}_n^\odot \mathcal{J}(-\mathbf{n}_m)), \end{aligned} \quad (10)$$

where we denote  $U_n = D^T \cdot \mathcal{B}(D \mathbf{V}_n)$  and  $\mathbf{V}_n = \text{Exp}(-\mathbf{n}_m) \text{Exp}(-\xi_m) \text{Exp}(\xi_s^i) \cdot E$  for notational convenience.

Equation (10) establishes the linear relationship between the perturbation in the pose and the magnetic field measurement.

Then, we can formulate the nonlinear system as a function  $h(\mathbf{n}_m)$ , where the pose perturbation  $\mathbf{n}_m$  serves as the estimation variable

$$B = h(\mathbf{n}_m) + \mathbf{n}_s, \quad \mathbf{n}_s \sim \mathcal{N}(\mathbf{0}, R_s) \quad (11)$$

where  $h(\cdot)$  is constructed by vertically stacking the measurement models of the  $N$  sensors in (9), and  $\mathbf{n}_s$  represents the vertical concatenation of  $\mathbf{n}_s^i$ .

Linearizing this model via a first-order Taylor expansion around the true state  $\mathbf{n}_0 = \mathbf{0}_{6 \times 1}$  yields

$$B \approx h(\mathbf{n}_0) + J_n(\mathbf{n}_0)(\mathbf{n}_m - \mathbf{n}_0) + \mathbf{n}_s \quad (12)$$

where  $J_n$  is formed by stacking  $J_n^i$  for  $i = 1, \dots, N$ .

Consequently, the objective function for the linearized model is formulated as follows:

$$f_n(\mathbf{n}_m) = \frac{1}{2} (\bar{y} - J_n(\mathbf{n}_0) \delta \mathbf{n}_m)^T R_s^{-1} (\bar{y} - J_n(\mathbf{n}_0) \delta \mathbf{n}_m) \quad (13)$$

where  $\bar{y} = B - h(\mathbf{n}_0) = \mathbf{n}_s$  is the measurement residual and  $\delta \mathbf{n}_m = \mathbf{n}_m - \mathbf{n}_0$ . Minimizing this function by equating its derivative to zero derives the optimal step

$$\delta \hat{\mathbf{n}}_m = \mathbf{G} \mathbf{n}_s, \quad \mathbf{G} := (J_n^T R_s^{-1} J_n)^{-1} J_n^T R_s^{-1}. \quad (14)$$

Thus, by propagating the covariance through the linear transformation, the measurement noise covariance can be propagated to the uncertainty of pose estimation, enabling quantitative assessment of the system's localization confidence

$$R_m = \mathbf{G} R_s \mathbf{G}^T = (J_n^T R_s^{-1} J_n)^{-1}. \quad (15)$$

This conclusion can also be readily generalized to scenarios involving multiple localization targets. Since the magnetic fields generated by multiple sources satisfy the principle of linear superposition, the derivation of the Jacobian matrix remains fundamentally unchanged. Consequently, uncertainty analysis for multiple targets can be achieved simply by stacking the relevant matrices and expanding the state dimensions accordingly.

#### E. Adaptability to Practical System Deviations

In practical applications, challenges such as imperfect magnet geometries, sensor heterogeneity, or even hardware malfunctions may arise, leading to measurements with varying noise magnitudes or outliers. While additional efforts are required to ensure robust magnetic localization under these conditions, the proposed pose estimation and uncertainty analysis framework remains applicable through appropriate hardware substitution or model adaptation. Specific discussions are presented as follows.

1) *Irregular Magnetic Sources*: Cylindrical magnets are widely adopted in research due to their ease of fabrication, high standardization, superior consistency, and cost-effectiveness. Should a magnet in use exhibit defects, it can be readily replaced with a permanent magnet possessing the same physical properties. For magnetic sources with other shapes or inherent defects, utilizing the magnetic dipole model to represent the field distribution beyond a distance of twice the minimum bounding sphere radius is a viable approach [28], as the elliptical integral-based model effectively degenerates to the dipole model in the

far field. Furthermore, multipole expansion can be employed to compensate for near-field distortions caused by complex geometries [29]. Detailed derivations regarding the magnetic dipole model and its gradients are also provided in Appendix A.

Another effective strategy involves the characterization and calibration of the specific magnetic source [30]. For instance, an irregular permanent magnet can be approximated as multiple virtual magnetic dipoles with distinct moments located at various positions [31]. By solving for optimal parameters using data points collected via grid scanning, the surrounding magnetic field of an irregular source can be accurately characterized. Alternatively, high-fidelity representation of the magnetic model can be achieved by applying Gaussian Process Regression, or grid interpolation combined with finite difference methods, directly to the scanned data [32], [33], [34].

We posit that replacing the elliptical integral-based model and its gradient with these alternative approaches covers the vast majority of geometry-related issues. In this article, without loss of generality, we focus on analyzing the elliptical integral model due to its high accuracy in the near field. However, the proposed methodology remains valid for other models, provided their gradient tensors are computable.

2) *Sensor Heterogeneity and Fault Tolerance*: Many commercially available sensors, such as the BMM150 (Robert Bosch GmbH) and MLX90393 (Melexis), exhibit significant discrepancies in noise levels and sensitivity between the in-plane axes ( $x/y$ ) and the vertical axis ( $z$ ) due to the nature of their manufacturing processes. Consequently, when employing hybrid sensor arrays or sensors with anisotropic noise characteristics, it is necessary to adjust the corresponding diagonal elements of the noise covariance matrix  $\mathbf{R}_s$  in (6) and (15). Since our approach utilizes MLE weighted by sensor noise, the estimator adaptively adjusts the confidence assigned to different measurement components based on the noise covariance matrix, thereby yielding a minimum-variance pose estimation.

Furthermore, in cases where a sensor malfunctions and produces outliers, or if a sensor is sensitive to only a single axis, its variance can be set to infinity (effectively assigning zero weight), thereby excluding it from the optimization process. In practical implementation, as described in Appendix B, a Chi-square ( $\chi^2$ ) test based on the Mahalanobis distance of residuals can be employed to diagnose the system and identify specific sensors with anomalous readings.

### F. Mitigate Tracking Uncertainty

Another goal of this work is to reduce the uncertainty in practical applications, thereby improving the accuracy of magnetic localization. As derived in (15), we have established the relationship between sensor noise and pose uncertainty. However, since this uncertainty originates in  $\mathfrak{se}(3)$ , it is not immediately intuitive. Thus, further clarification is required to justify the use of the covariance matrix  $\mathbf{R}_m$  as a metric for uncertainty evaluation and to explain its advantages.

In the SE(3) manifold, perturbations at a given point can be naturally represented as vectors in the tangent space at that point, which facilitates covariance representation. As shown in

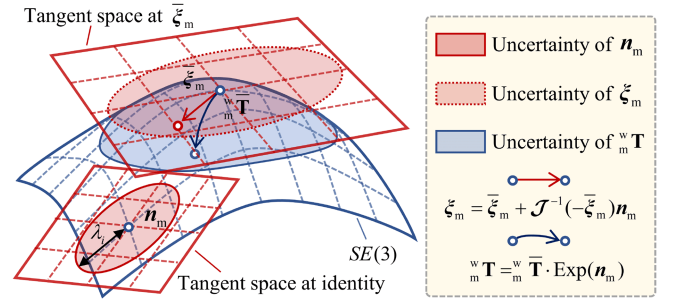


Fig. 3. Pose uncertainty of the localized magnet. The pose perturbation  $\mathbf{n}_m$  is a zero-mean random variable defined in the tangent space at the identity of SE(3). It is transported to the tangent space at the estimated pose  $\bar{\xi}_m$  via the inverse of the left Jacobian  $\mathcal{J}$ , and then mapped onto the manifold through the exponential map to represent the uncertainty on SE(3).

Fig. 3, we visualize the uncertainty ellipsoid defined by  $\mathbf{R}_m$  in the tangent space at the identity. This zero-mean noise  $\mathbf{n}_m$  is then mapped to the tangent space at the estimated pose  $\bar{\xi}_m$  via the inverse of the Jacobian, and subsequently pushed forward to the manifold SE(3) via the exponential map. The procedure for transferring the noise from the tangent space at the identity to that at the target pose is as follows:

$$\bar{\xi}_m = \bar{\xi}_m + \mathcal{J}(-\bar{\xi}_m)^{-1} \mathbf{n}_m, \quad \text{if } \mathbf{n}_m \text{ small.} \quad (16)$$

As depicted in Fig. 3, three kinds of uncertainty are involved. First, the covariance ellipsoid of  $\mathbf{n}_m$  is visualized in the tangent space at the identity. This ellipsoid is transformed via (16) into the dashed red region, which represents the uncertainty of the pose  $\xi_m$  in the Lie algebra. Importantly, because the transformation is nonlinear, the resulting region is no longer an ellipsoid in the strict sense. Finally, the uncertainty of  $\xi_m$  is mapped to SE(3) as the uncertainty of the transformed pose  $\xi_m^w T$ .

Notably, we do not use the result from (8), which provides a linearized relationship between sensor measurement and pose mean to directly compute the uncertainty of  $\xi_m$ . Instead, we deliberately introduce the perturbation term  $\mathbf{n}_m$  for a key reason: the rotation it induces is applied in the local frame, where different axes remain decoupled. This is particularly important for common magnetic targets, such as cylindrical magnets or solenoids, whose magnetic fields are often axisymmetric. Consequently, the rotational degree of freedom around the magnet's own  $z$ -axis has negligible impact on the measured field. By simply zeroing the last entry of the covariance matrix  $\mathbf{R}_m$ , we can exclude the rotation about the symmetry axis from uncertainty analysis. In contrast, such decoupling is not possible if we use (8) directly for uncertainty propagation.

Thus, only the uncertainty region for  $\mathbf{n}_m$  in Fig. 3 is a standard hyper-ellipsoid ellipsoid that enables convenient volume computation [35]. The volume of the covariance ellipsoid defined by  $\mathbf{R}_m$  is proportional to the product of its eigenvalues and is given by

$$\text{Vol}(\mathbf{R}_m) = \frac{\pi^{n/2}}{\Gamma(1+n/2)} \prod_{i=1}^n \lambda_i = \frac{\pi^{n/2}}{\Gamma(1+n/2)} \det(\mathbf{R}_m) \quad (17)$$

where  $\Gamma(\cdot)$  is the gamma function and  $n$  represents the number of DoF involved in modeling the pose uncertainty. In this study, we consider  $n = 5$  for a cylindrical magnet.

The volume of the uncertainty ellipsoid is adopted as the primary localization metric because it provides a coordinate-invariant scalar representation of the total confidence region. While individual eigenvalues or condition numbers can characterize specific aspects of the covariance matrix, they lack direct physical interpretability in our SE(3)-based framework due to the inherent coupling between translational and rotational perturbations. Furthermore, within the defined effective workspace, the balanced constraints of the magnetic field gradient across space ensures a stable Jacobian condition number, meaning the volume reliably reflects the overall precision without being skewed by extreme directional anisotropy.

For the magnetic sensor array proposed in this work that is movable in the  $x - y$  plane, reducing the localization uncertainty is cast as the following optimization problem:

$$\begin{aligned} \min \quad & f(x_s, y_s) = \text{Vol}(\mathbf{R}_m) \\ \iff \min \quad & g(x_s, y_s) = -\det(\mathbf{J}_n^T \mathbf{R}_s^{-1} \mathbf{J}_n) \end{aligned} \quad (18)$$

where  $x_s$  and  $y_s$  represent the position of the sensor board along the  $x$  and  $y$  directions. The equivalent optimization problem in (18) can be solved using various optimization algorithms.

We have only introduced 2 DoF for the magnetic sensor array and deliberately avoided introducing mechanisms, such as radar-like rotational scanning for two practical reasons. First, in medical scenarios, the patient often lies in close proximity to the sensor array, leaving no room for rotational scanning. Second, proximity between the magnet and the sensor generally reduces uncertainty. Therefore, the optimization in (18) would undesirably drive the sensor board to coincide spatially with the magnet, which is impractical and physically meaningless. Hence, in this study, we restrict the motion of the array to the  $x - y$  plane only. As demonstrated in the subsequent experiments, this design is already sufficient for typical application scenarios.

### III. SIMULATION ANALYSIS

To evaluate the proposed approach and localization system prior to physical deployment, we conduct a series of simulations under varying conditions. These simulations are designed to visualize the uncertainty associated with magnetic localization and to assess the effective workspace of the magnetic sensor array. Furthermore, we investigated how the orientation of the target magnet, as well as the number and configuration of magnetic sensors, influence localization accuracy and workspace coverage.

Unless otherwise specified, all subsequent simulations adopt a cylindrical magnet with radius  $R = 5$  mm and half-length  $L = 5$  mm, magnetized to a typical value of  $10^6$  A/m (comparable to N52 magnets). The sensor noise covariance is modeled as  $\mathbf{R}_s^i = \text{Diag}(1\mu\text{T}^2, 1\mu\text{T}^2, 1\mu\text{T}^2)$ , consistent with the RMS noise levels reported in commercial magnetic sensor datasheets.

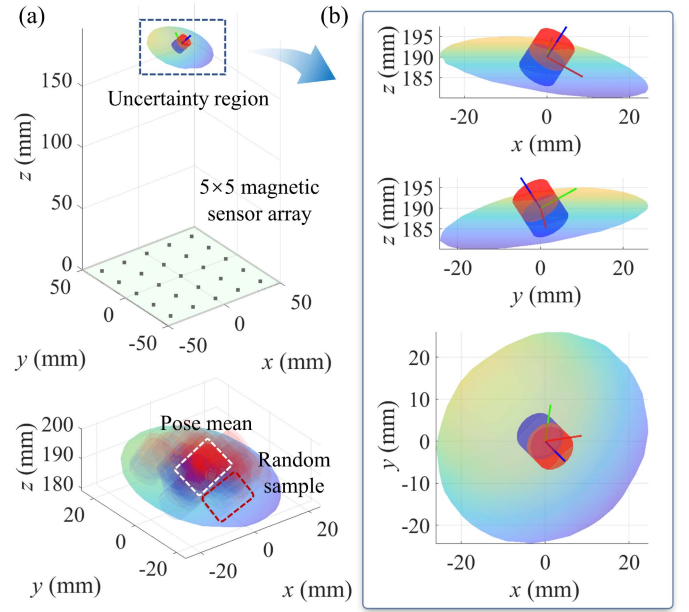


Fig. 4. Visualization of position and orientation uncertainty of the magnet localized by the sensor array in Cartesian space. (a) Relative pose between the magnet and the sensor array, along with random sampling of localization results. (b) Three orthographic views of the uncertainty surface.

#### A. Uncertainty of Pose

We first quantify localization uncertainty using a  $5 \times 5$  magnetic sensor array with 20 mm spacing. The uncertainty propagation from sensor noise to pose disturbance is computed using (15). To facilitate visualization of the resulting pose uncertainty in Cartesian coordinates, we placed the magnet directly above the center of the array at a height of 190 mm, with an initial orientation vector set to  $[\frac{\pi}{6}, \frac{\pi}{6}, 0]^T$  as shown in Fig. 4(a).

Due to the 5 DoFs in the magnet's pose, the uncertainty ellipsoid in Cartesian space is generally not available. Instead, we sampled a large number of pose perturbations from the 95% confidence region defined by the covariance matrix  $\mathbf{R}_m$ , and then mapped these perturbations to SE(3) via the matrix exponential. This yielded a dense distribution of perturbed magnet poses in Cartesian space. By constructing an envelope surface around all sampled positions, we obtained a surface representation of the spatial pose uncertainty, shown in Fig. 4(b).

Notably, the resulting uncertainty region does not exhibit a standard ellipsoidal shape. As a result of the nonlinearity introduced by the exponential map as in (1), the cross-sections of the 3-D distribution adopt a characteristic “banana-shaped” profile, as described in [36], which effectively characterizes the spatial region, where the magnet is most likely to be located. In the remainder of this article, we refer to this envelope as the uncertainty surface.

In addition, since the magnetic field distribution around the magnet varies with its orientation, the measurement noise perceived by the sensors will also change accordingly. As a result, even at the same spatial location, the localization uncertainty may differ significantly for different magnet orientations. Leveraging the proposed uncertainty estimation method, we can



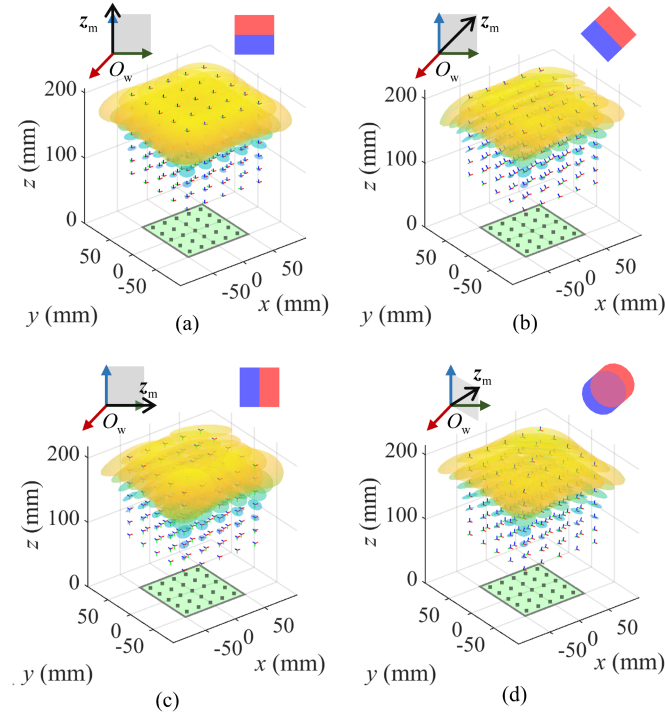


Fig. 5. Visualization of localization uncertainty for different magnet orientations. (a)  $\mathbf{z}_m = [0, 0, 1]^T$ . (b)  $\mathbf{z}_m = [0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^T$ . (c)  $\mathbf{z}_m = [0, 1, 0]^T$ . (d)  $\mathbf{z}_m = [\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}]^T$ .

quantitatively evaluate the localization uncertainty for various positions and orientations of the magnet.

As shown in Fig. 5, we selected four representative magnet orientations and computed the uncertainty at grid points located within a region extending 120 mm  $\times$  120 mm laterally and from 80 to 200 mm in height above the sensor array. The results clearly demonstrate that localization uncertainty at a fixed position varies substantially with magnet orientation, as reflected in both the shape and the volume enclosed by the uncertainty surface. In Fig. 5(a) and (c), the magnetization vectors are aligned along the  $z$ -axis and  $y$ -axis, respectively. In these cases, the magnetic field is symmetrically distributed in the sensor array frame, leading to nearly symmetric uncertainty surfaces. Moreover, the uncertainty surfaces become larger as the magnet moves closer to the edges of the sensor array, which aligns with intuitive expectations.

In contrast, when the magnetization vector is tilted at an angle of  $\frac{\pi}{4}$ , as in Fig. 5(b) and (d), the uncertainty surfaces become asymmetric in space. To further quantify this effect, Fig. 6(a) shows the uncertainty variation with respect to the magnet's orientation angle about the  $x$ -axis (from 0 to  $\pi$ ) at three points—A, B, and C—located at the same height (150 mm) above the array. The uncertainty is computed using (6). A logarithmic scale is employed to compress the value range and highlight relative differences.

In the following analysis, we adopt  $\log(\text{Vol}(\mathbf{R}_m))$  as the primary indicator of uncertainty magnitude. As shown in Fig. 6(b), point B, which is directly above the center of the sensor array exhibits the lowest uncertainty overall. Furthermore, as the

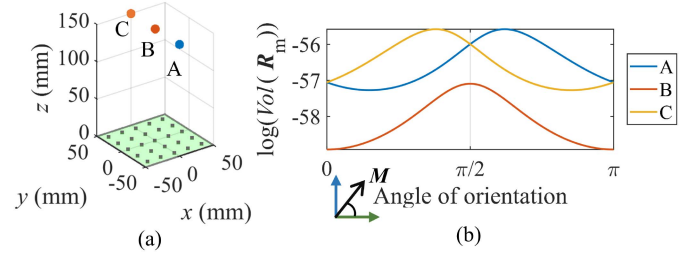


Fig. 6. Variation of localization uncertainty with magnet orientation angle. (a) Three representative points (A, B, and C) located above the sensor array. (b) The localization uncertainty exhibits periodic fluctuations as the magnet orientation changes.

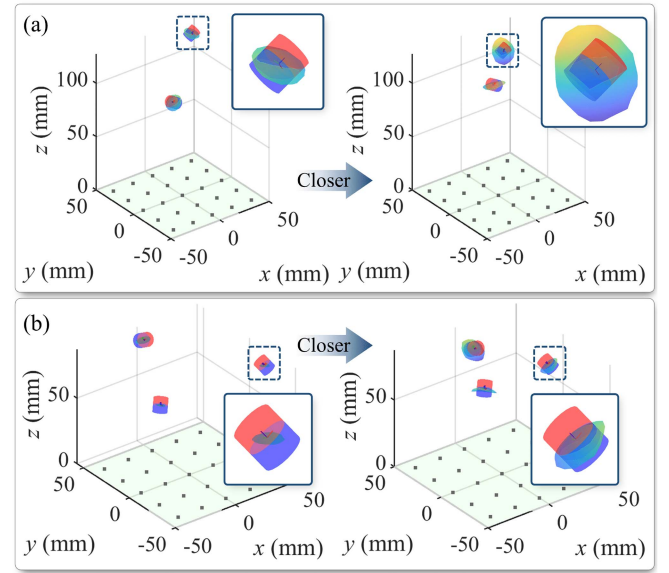


Fig. 7. Localization uncertainty of multiple target magnets increases as the inter-magnet distance decreases. (a) Two magnets. (b) Three magnets.

magnet orientation changes, the uncertainties at these points vary periodically, indicating a strong dependency of localization uncertainty on magnet orientation.

For multimagnet localization, as noted above, the superposition property of magnetic fields allows us to form the overall Jacobian  $\mathbf{J}_n$  by horizontally concatenating the Jacobian blocks associated with individual magnets. Substituting  $\mathbf{J}_n$  into (15) yields an enlarged covariance matrix for the concatenated pose perturbations of all magnets. This matrix is generally not block-diagonal, i.e., the cross-covariances between the pose perturbations of different magnets are nonzero, because their poses are estimated through a magnetically coupled measurement model.

We simulated the localization uncertainty for the cases of two and three magnets, as shown in Fig. 7. In contrast to the single-magnet case, where uncertainty decreases as the magnet approaches the center of the sensor array, reducing the inter-magnet distance at the same height leads to a sharp increase in uncertainty, even though each magnet becomes closer to the array center. We attribute this behavior to reduced identifiability when magnets are in close proximity, while the magnetic field contributions add linearly, the information content is not linearly separable. From the Jacobian perspective, decreasing the

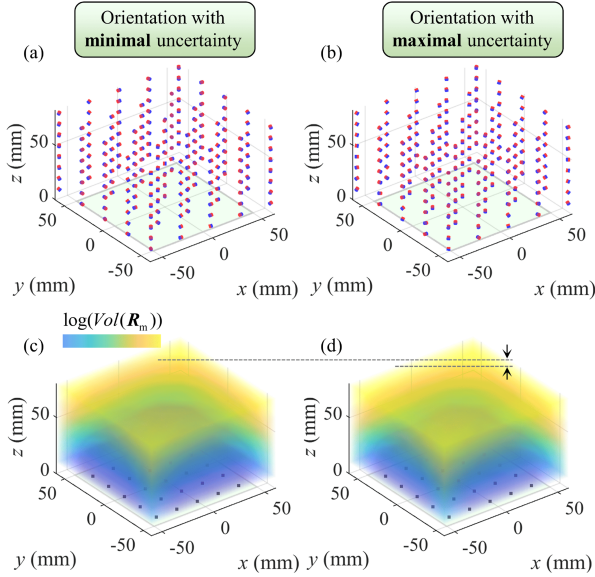


Fig. 8. Influence of magnet orientation on the effective localization workspace. (a) Magnet orientation that minimizes localization uncertainty at each point. (b) Orientation that maximizes uncertainty. (c) and (d) Spatial distributions of uncertainty corresponding to (a) and (b), respectively.

intermagnet distance increases the column (block) correlation in  $\mathbf{J}_n$ , potentially rendering  $\mathbf{J}_n^T \mathbf{J}_n$  ill-conditioned, which in turn inflates the estimated covariance. Therefore, this insight suggests that in applications requiring multitarget localization, the magnets should be separated by a sufficient distance to achieve higher accuracy. For example, in prior studies, muscle contraction is monitored by tracking intermagnet distance changes to control prosthetic devices [12], [13].

To isolate the fundamental factors governing localization uncertainty and provide clear design guidelines, we focus on the single-magnet case in the subsequent analysis. While the proposed framework naturally extends to multiple magnets by concatenating the corresponding Jacobian blocks, multimagnet localization introduces nontrivial coupling effects and potential identifiability degradation when magnets are in close proximity, which makes the resulting uncertainty highly configuration-dependent. Therefore, we use the single-magnet setting as the canonical scenario to derive interpretable trends and actionable insights.

### B. Workspace Analysis

Having established a quantitative metric for uncertainty assessment, we are now able to perform a priori analysis of the effective workspace of a magnetic localization system. This includes evaluating the workspace's size, geometry, and spatial uniformity, which in turn informs the selection of appropriate sensor array configurations. Such analysis enables better confidence in the localization performance in practical applications.

To explore this, we uniformly sampled grid points within the space above the sensor array and, at each point, computed the magnet orientation that minimizes and maximizes the localization uncertainty, respectively. The resulting optimal and worst-case orientations are shown in Fig. 8(a) and 8(b). Based on

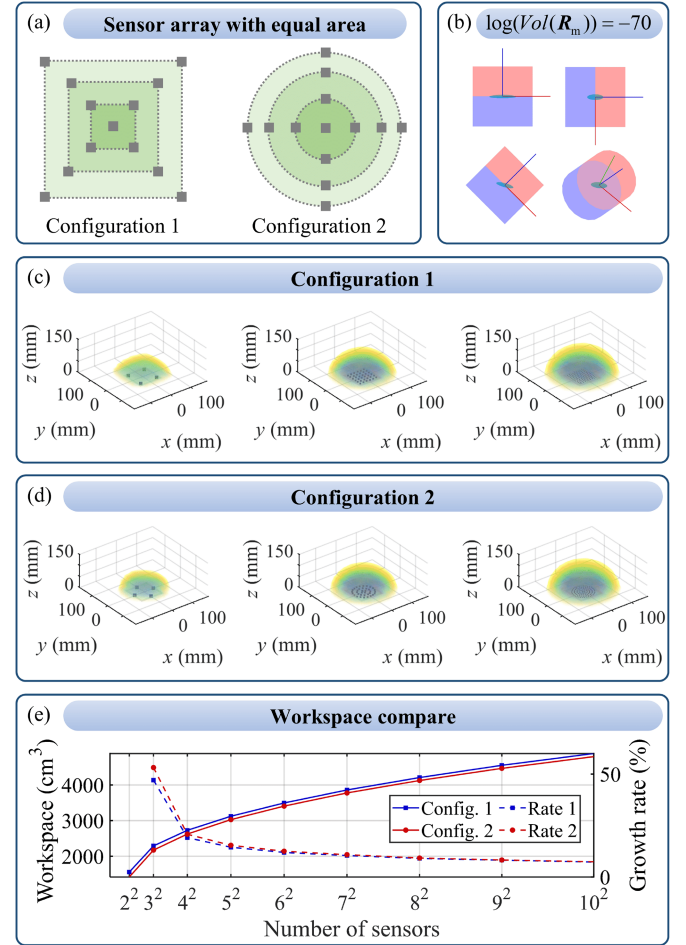


Fig. 9. Analysis of the effective workspace for different sensor array configurations. (a) Square and circular arrangements with equal enclosed areas and evenly distributed sensors. (b) Visualization of the uncertainty surface when the uncertainty metric equals  $-70$ . (c) and (d) Effective workspaces for square and circular arrays with 4, 36, and 100 sensors, respectively. (e) Variation in workspace size and growth rate with increasing sensor count for both array types.

these orientations, we computed the corresponding localization uncertainty across the entire grid and visualized the results in Fig. 8(c) and (d) using color-coded spatial mappings. It can be observed that for a given color level, the uncertainty values under the orientation in Fig. 8(a) are slightly higher than those under the orientation in Fig. 8(b), with a typical difference of approximately 3 mm.

Therefore, in this study, we conservatively adopt the worst-case orientation as the indicator for workspace evaluation. This provides a guaranteed lower bound on system performance, meaning that for any other magnet orientation, the uncertainty will be no worse than the distribution shown in Fig. 8(d).

Based on the preceding analysis, we further investigate how the workspace is influenced by sensor arrangement and quantity, aiming to identify more optimal array configurations. We consider two kinds of sensor arrangement configurations: square and circular.

As illustrated in Fig. 9(a), for a fair comparison, the square and circular sensor arrays are designed to occupy the

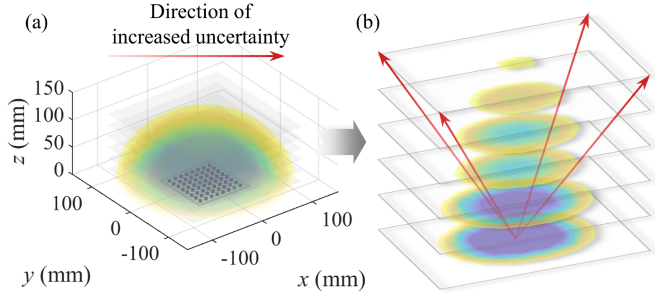


Fig. 10. Direction of localization uncertainty increase within the sensor array workspace. (a) Workspace visualization for a square array with 64 sensors. (b) Cross-sectional slices showing uncertainty distributions at different heights.

same total area, with an equal number of sensors distributed along each concentric ring. We define an uncertainty metric  $\log(\text{Vol}(\mathbf{R}_m)) \leq -70$  as effective for localization. The corresponding uncertainty surface [see Fig. 9(b)] indicates that the uncertainty volume does not exceed 3 mm. The effective workspaces of the square and circular arrays are shown in Fig. 9(c) and (d), respectively. Both configurations yield workspaces of similar shape, and for the same number of sensors, the difference in workspace volume is relatively minor.

However, quantitative analysis in Fig. 9(e) reveals that the square configuration consistently produces a slightly larger workspace than the circular one. This aligns with the common preference for square layouts in most magnetic localization systems reported in the literature. Moreover, as the number of sensors increases, the effective workspace expands accordingly. Nevertheless, as indicated by the dashed lines in Fig. 9(e), the rate of workspace growth diminishes with sensor count, revealing a clear marginal diminishing effect. These findings suggest that simply increasing the number of sensors within a fixed area does not indefinitely improve workspace coverage. Therefore, for optimal array design, rather than simply increasing sensor density, resources should be prioritized towards optimizing the array's geometric diversity relative to the expected source trajectory to maximize the information gain per sensor, which can be quantified by evaluating the ratio of the uncertainty ellipsoid volumes before and after adding a specific sensor along a standard trajectory.

### C. Uncertainty Mitigation

The aforementioned analytical framework not only enables effective evaluation of the workspace of magnetic localization systems, but also serves as a theoretical foundation for actively reducing localization uncertainty during system operation. In most practical scenarios, the position of the target to be localized is uncontrollable. Consequently, improving localization accuracy must rely on dynamically adjusting the position of the sensor array, making the development of an optimal repositioning strategy critically important.

As illustrated in the cross-sectional slices of Fig. 10, localization uncertainty increases rapidly with both the height above the sensor array and lateral displacement from its center. This observation is intuitive and suggests that, whenever possible,

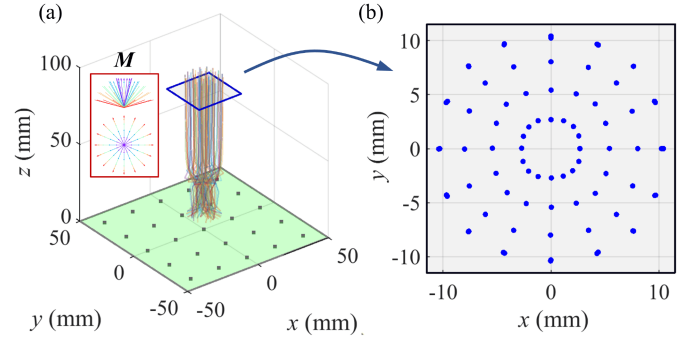


Fig. 11. Magnet orientation-dependent optimal positions for minimizing localization uncertainty at different heights. (a) Trajectories of minimum-uncertainty positions across various heights for fixed magnet orientations. (b) The cross-sectional view of these trajectories forms a circular scatter pattern, whose spread and orientation are determined solely by the magnetic moment direction.

the projection of the target magnet should remain near the center of the sensor array. However, as previously discussed, the orientation of the magnet also significantly affects the shape and size of the resulting uncertainty surface. This implies that the point directly above the sensor center is not necessarily the location of minimal uncertainty.

This insight is further validated in Fig. 11. We systematically sampled a range of magnetic moment directions, as shown by the red bounding box in Fig. 11(a), and for each orientation, we identified the location at each height where the localization uncertainty reaches its minimum. The resulting trajectories of minimal-uncertainty positions reveal that the optimal relative position between the magnet and the sensor array depends strongly on the magnet's orientation. Interestingly, for a given magnet orientation at a fixed height, the optimal position in the  $xy$ -plane remains constant. That is, the direction and distance of this optimal offset relative to the sensor center depend only on the orientation of the magnetic moment.

This relationship is visualized in Fig. 11(b), where the collection of these optimal offsets forms a circular scatter pattern that directly reflects the orientation of the magnetic moment. Notably, we conduct extensive numerical simulations and confirm that the diameter of this scatter pattern is not affected by sensor noise, magnetization strength, or the physical size of the magnet, but is instead solely determined by the configuration of the sensor array.

Therefore, for a given sensor array configuration, the optimal sensor position corresponding to each magnet orientation can be precomputed using the optimization objective defined in (18), without the need for solving this problem online during real-time localization. This significantly improves computational efficiency and system responsiveness.

To demonstrate the effectiveness of the proposed uncertainty analysis and the uncertainty reduction strategy in magnetic localization, we conducted a simulated magnetic trajectory tracking experiment, as illustrated in Fig. 12. A 3-D helical trajectory is designed, starting directly above the center of the sensor array, with the height (ranging from 30 to 130 mm) and radial distance (from 25 to 125 mm) increasing linearly. The



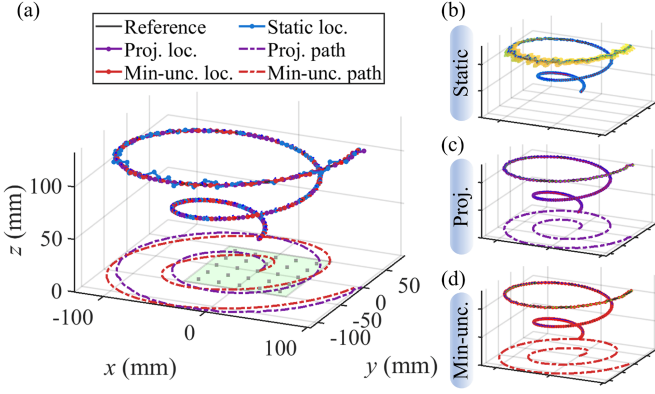


Fig. 12. Simulation results of helical trajectory tracking using different localization strategies. (a) Tracking results for the static sensor array, projection-based array movement, and the proposed min-uncertainty strategy. (b) and (c) Visualization of localization uncertainty along the trajectory under the three strategies. Abbreviations: Ref., reference; Proj., projection-based; Min-unc., min-uncertainty; Loc., localization.

orientation vector along the trajectory is designed to vary periodically based on trigonometric functions. During the simulation, zero-mean Gaussian noise with covariance  $\mathbf{R}_s^i$  is added to the triaxial magnetic field measurements at each sensor.

Three kinds of strategies are compared as follows.

- 1) *Static*: the sensor array remains fixed.
- 2) *Projected*: the sensor array follows the projection of the magnet's position onto the array plane.
- 3) *Min-Uncertainty*: the array dynamically moves to minimize the localization uncertainty, as in (18).

As shown in Fig. 12(a), the difference among the strategies is negligible at low altitudes and small radii, where the magnet remains within the core sensing region. However, as the height and radius increase and the magnet moves beyond the original effective sensing area, the static sensor array fails to maintain accurate tracking, resulting in significant errors. In contrast, both the projection-based and the min-uncertainty strategies succeed in maintaining acceptable tracking performance. Notably, the trajectory of the min-uncertainty array exhibits a compressed spiral shape. This arises due to the sinusoidal variation in the magnet's orientation, which causes the optimal sensor position to move alternately closer to and farther from the projection curve and is consistent with the analysis in Fig. 11.

Fig. 12(b)–(d) presents the spatial distribution of localization uncertainty along the trajectory for all three strategies. The static strategy clearly leads to significantly higher uncertainty, especially in the latter part of the trajectory. In contrast, the projection-based and min-uncertainty strategies maintain low levels of uncertainty throughout, as further quantified in Fig. 13(a).

While Figs. 12(a) and 13(a) may not visually reveal a substantial difference between the projection-based and min-uncertainty strategies, from a theoretical perspective, the min-uncertainty method outperforms the projection-based approach. This is because it explicitly minimizes the expected pose perturbation in both position and orientation.

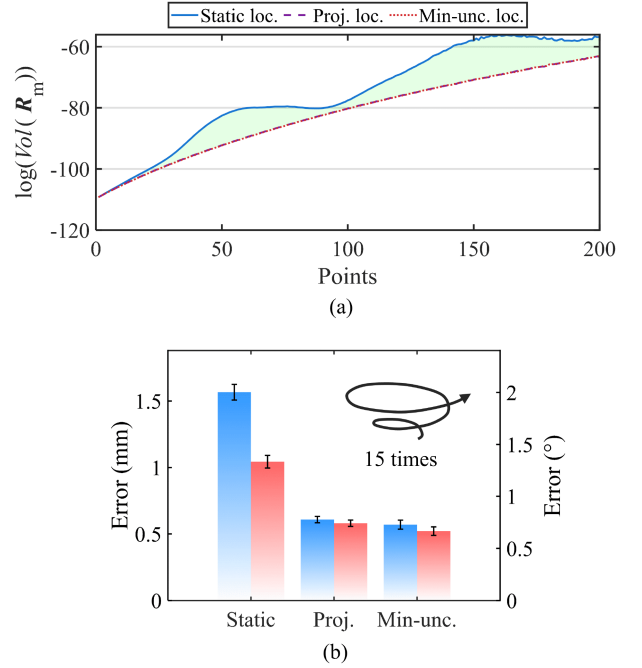


Fig. 13. Comparison of uncertainty and localization errors during helical trajectory tracking. (a) Uncertainty magnitude along the trajectory. (b) The mean positional error and orientation error of trajectory tracking under the three localization strategies, evaluated across multiple simulated trajectories.

This theoretical advantage is corroborated by the quantitative results shown in Fig. 13(b). We adopt the following error metrics to evaluate the accuracy of the position and orientation estimate at the test point

$$e_p = \|\mathbf{p}_{\text{ref}}^w - \mathbf{p}_{\text{test}}^w\|_2 \quad (19a)$$

$$e_o = \cos^{-1}(\mathbf{e}_z^{\text{T}w} \mathbf{R}_{\text{ref}}^{\text{T}w} \mathbf{R}_{\text{test}}^w \mathbf{R} \mathbf{e}_z) \quad (19b)$$

where  $\mathbf{e}_z$  denotes the unit vector  $[0, 0, 1]^T$ , with the subscripts *ref* and *test* representing the ground truth and the estimated values, respectively. This definition of orientation error is adopted because the cylindrical magnet lacks a degree of freedom for rotation about its own  $z$ -axis; therefore, only the orientation of its magnetic moment is of our interest.

The average position error and orientation error for the static strategy are 1.56 mm and 1.33°, respectively. For the projection-based method, these errors are reduced to 0.61 mm and 0.74°, while the min-uncertainty strategy achieves the lowest errors of 0.57 mm and 0.66°. This corresponds to a 60.90% and 6.56% reduction in position error, and a 44.36% and 10.81% reduction in orientation error compared to the static and projection-based strategies, respectively.

#### IV. EXPERIMENTS

To further demonstrate the effectiveness of the proposed pose estimation, uncertainty analysis, and reduction strategy, we developed a magnetic localization system equipped with 2 DoFs for planar motion. The advantages of the proposed method are validated through both system characterization and comparative experiments. In addition, we evaluate its practical applicability

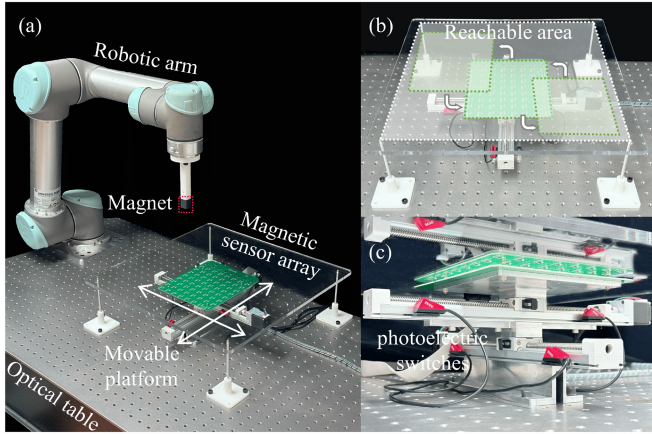


Fig. 14. Experimental setup for dynamic magnetic localization and uncertainty analysis. (a) The system consists of a robotic arm holding a N52 magnet and a magnetic sensor array mounted on a two-DOF movable platform. (b) Reachable area of the planar sensor array. (c) Configuration of the orthogonally mounted linear modules, with stroke constrained by photoelectric switches.

in phantom experiments simulating medical scenarios, including the alignment of locking screws in interlocking nail procedures and the localization of magnetic capsules. These experiments highlight the potential to improve the localization accuracy of conventional fixed magnetic systems in clinical applications.

#### A. Experimental Setup

As shown in Fig. 14(a), the experimental system primarily consists of a robotic arm (Universal Robots, Denmark) holding an N52-grade neodymium magnet, a movable platform composed of two orthogonally mounted miniature linear modules (FSK30 J, FUYU Technology Co., Ltd., China), and a planar magnetic sensor array comprising 64 three-axis magnetometers (LS2MDL, STMicroelectronics, Switzerland) arranged in an  $8 \times 8$  grid with 20 mm spacing. The sensor array is mounted on the upper linear module, allowing planar motion with two DoF. The platform driven by stepper motors provides a maximum speed of 80 mm/s and a repeatability of  $\pm 0.05$  mm, which is sufficient for localizing low- to medium-speed targets. By reading the stage position at a high update rate, we minimize the position-update gaps of the moving platform and thereby mitigate potential motion artifacts in dynamic localization. The reachable area of the array is illustrated in Fig. 14(b), and is primarily determined by the stroke of the linear modules. In this setup, the maximum travel range is limited to 160 mm by photoelectric switches, as shown in Fig. 14(c).

On the software side, the 64 magnetometers are interfaced via an SPI bus and read sequentially by an Arduino Mega 2560 (Arduino, Italy), achieving a data acquisition rate of 100 Hz. The same Arduino unit also controls the linear modules and reads their current positions in real time. The robotic arm is connected to a host computer via Ethernet and controlled using the real-time data exchange interface for trajectory planning and path execution. The tracking algorithm is implemented in C++ for real-time performance. Magnet poses are resolved using the gradient-based *lmdr1* function from the CMath library, while

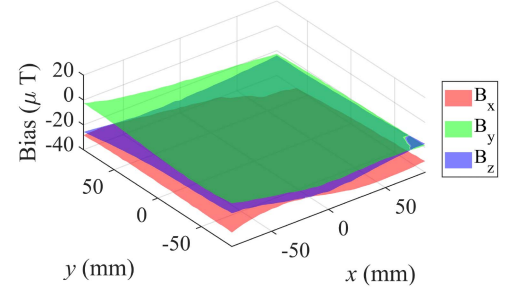


Fig. 15. Triaxial measurement biases of a representative sensor within the array across the entire workspace, accounting for both the ambient geomagnetic field and intrinsic sensor offsets.

the uncertainty minimization is handled by NLOpt. A Qt-based GUI is integrated to facilitate human-computer interaction and real-time visualization.

Due to the presence of the geomagnetic field (ranging from approximately 25–65 uT at the surface) and the intrinsic offsets of the sensors, the raw readings cannot be directly utilized in the optimization algorithm. During the experimental preparation phase, the movable sensor array was operated to traverse the entire workspace, as depicted in Fig. 14(b). The triaxial magnetic data were recorded at each point to construct a background magnetic field map. By employing linear interpolation and a real-time look-up table, the biases of the measurements is compensated before each optimization step. This procedure ensures that the input data for the localization algorithm is exclusively the field generated by the external source. The triaxial background biases measured in the experimental site are shown in Fig. 15.

#### B. Magnetic Localization With Different Orientations

We first conduct localization experiments with the magnetic source in two distinct orientations, while the system remains stationary, aiming to evaluate the practical impact of sensor noise on localization accuracy. Two representative magnetic moment orientations are selected:  $\mathbf{z}_m = [0, 0, 1]^T$  and  $\mathbf{z}_m = [0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^T$ . The magnet, held by the robotic arm, is scanned at a height of 40 mm above the sensor array, covering a 3-D volume of 160 mm  $\times$  160 mm  $\times$  120 mm, and the test points are uniformly spaced at 10 mm intervals.

After subtracting the precalibrated sensor biases and the ambient magnetic field, we estimate the magnet's pose by solving (6). The ground truth is obtained from the forward kinematics of the robotic arm, and the position and orientation errors at each test point are computed using (19).

To visualize the spatial distribution of the localization errors, we employ trilinear interpolation to generate 3-D heatmaps of both the position and orientation errors for the two magnet orientations as shown in Fig. 16. Notably, under the same orientation, the heatmaps of position and orientation errors exhibit similar spatial patterns. This is because sensor noise simultaneously affects both position and orientation estimates, resulting in the same darker regions both in position error and orientation error.

Interestingly, the error distributions differ significantly between the two orientations. In Fig. 16(a) and (b), where the

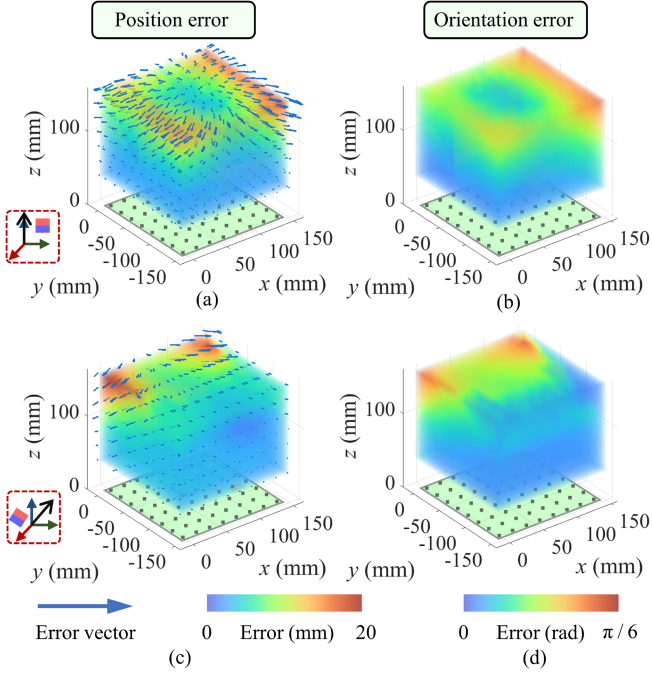


Fig. 16. Heatmaps of localization errors for magnets with different orientations. (a) and (b) Position and orientation errors for  $\mathbf{z}_m = [0, 0, 1]^T$ . (c) and (d) Position and orientation errors for  $\mathbf{z}_m = [0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^T$ .

magnetic moment vertically upward, the error distributions are approximately radially symmetric, with larger errors near the boundaries of the sensor array. In contrast, Fig. 16(c) and (d) shows that when the magnet is rotated by  $\frac{\pi}{4}$  about the negative  $x$ -axis, the errors become substantially larger on the positive  $y$ -axis side compared to the negative side. This observation is consistent with our uncertainty analysis in Section III-A, where, for instance, point C (see Fig. 6) exhibits significantly higher uncertainty than point A when the angle of orientation is  $\frac{\pi}{4}$ .

Moreover, in Fig. 16(a) and (c), we plotted error vectors at each test point, with direction from the ground truth to the estimated position and magnitude equals to the localization error. The distribution of these vectors, taking the top layer of test points as an example, closely resembles the shape of the uncertainty surfaces shown in Fig. 5(a) and (b). This correspondence further validates the effectiveness of the method for propagating sensor noise over SE(3) in analyzing the uncertainty characteristics of magnetic localization.

### C. Comparison of Static and Dynamic Array

We then validate the effectiveness of the planar 2-DoF magnetic localization platform capable of dynamically tracking the target, minimizing localization uncertainty and enlarging the effective workspace.

A high-level workflow of the system's operation is illustrated in Fig. 17, organized according to functional modules. The movable sensor array performs two main tasks: reading sensor data and moving the array to desired positions based on localization results. The array, comprising 64 three-axis magnetometers, samples data at approximately 100 Hz, while

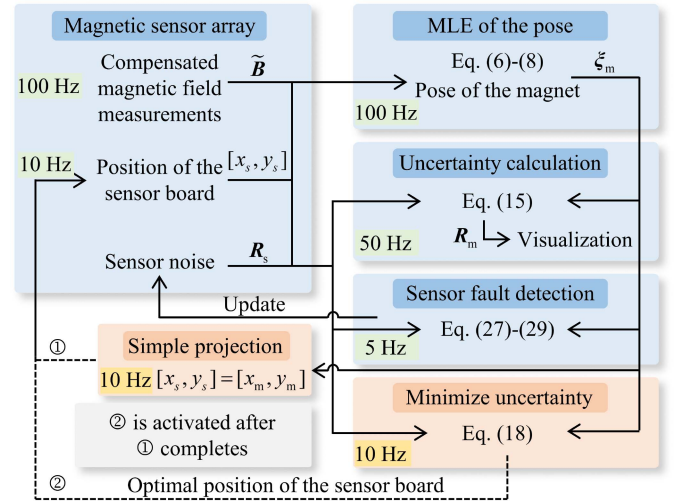


Fig. 17. Magnetic tracking workflow using a planar movable sensor array. To improve efficiency during magnet localization, the sensor array first rapidly moves beneath the magnet using a projection-based approach ①. Then, the minimum-uncertainty algorithm is iteratively applied to localize the magnet and update the sensor position until convergence is achieved ②.

due to speed constraints of the linear stages, the update rate for sensor movement is limited to 10 Hz.

Upon receiving raw magnetic field data from all 64 sensors, the ambient field is compensated before  $\tilde{\mathbf{B}}$  is fed into the magnetic localization module to estimate the magnet's pose  $\xi_m$  in the world frame  $\{w\}$ . This module operates at around 100 Hz on a standard office laptop. The estimated pose is then simultaneously used for visualizing uncertainty and computing the optimal displacement of the sensor array, enabling it to tracking the magnet in real time.

To improve responsiveness, both the projection-based and the min-uncertainty strategies are incorporated into the workflow. Initially, when the magnet is far from the sensor array's center, the system uses the projection-based method (① in Fig. 17) to quickly reposition the array directly beneath the estimated magnet location. Subsequently, iterative pose estimation and min-uncertainty optimization (② in Fig. 17) are performed until convergence is reached: both the pose estimation difference and sensor displacement falling below predefined thresholds.

Based on this workflow, we conduct trajectory-tracking experiments under three localization strategies: static, projection-based, and min-uncertainty. Two 3-D reference trajectories are designed: a five-pointed star and a helical path. The five-pointed star spans a height range of 113–134 mm and projects entirely within the sensor array, with a fixed magnet orientation throughout. In contrast, the helical trajectory spans 80–100 mm in height, with part of its projection lying outside the sensor array's coverage and features periodic changes in the magnet's orientation around its  $x$ -axis. These trajectories are chosen to enable fair comparisons of the three localization strategies in terms of position and orientation accuracy while highlighting how varying magnet orientations influence the motion patterns of the sensor array under the min-uncertainty strategy.

Tracking results under the three strategies are shown in Fig. 18. For the five-pointed star, the average position/orientation



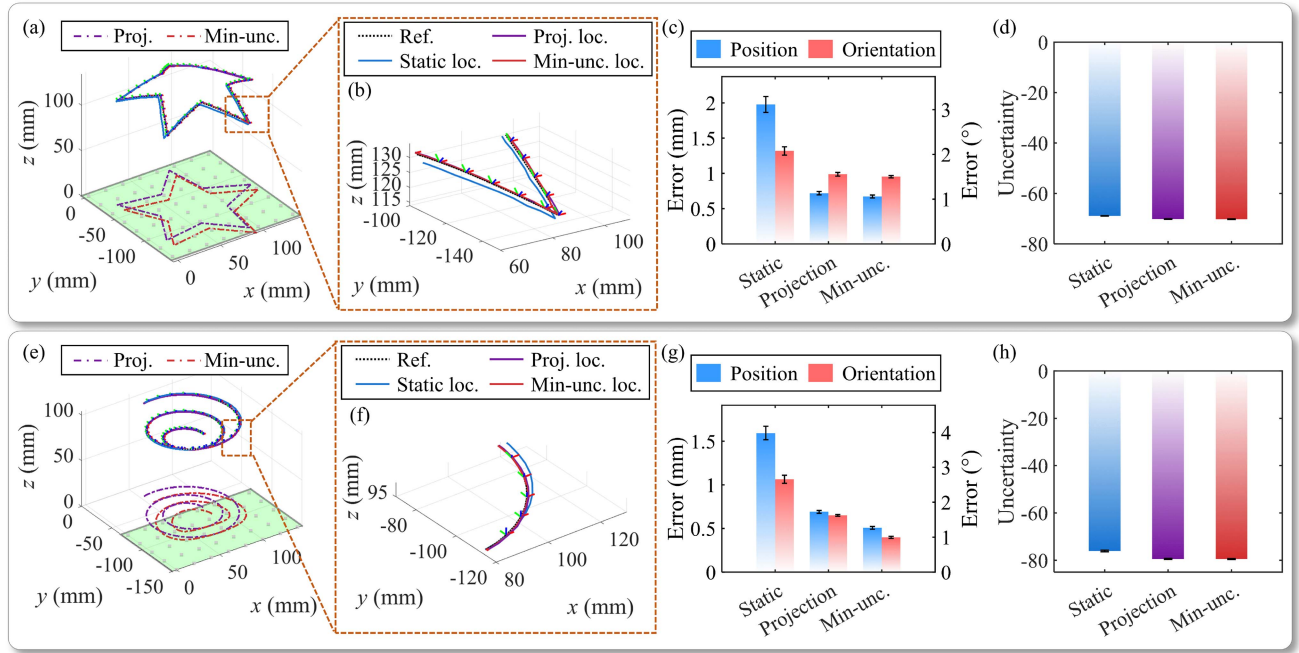


Fig. 18. Performance comparison of static, projection-based, and minimum-uncertainty tracking on custom-designed trajectories. (a) Localization results on the spatial five-pointed star trajectory. Two dashed lines on the sensor array indicate its movement; since the magnet orientation is fixed along the trajectory, the shapes of the sensor trajectories under projection-based and min-uncertainty strategies are nearly identical, differing only in translation. (b) Zoomed-in view of the five-pointed star trajectory tracking results. (c) Position and orientation errors along the five-pointed star trajectory. (d) Uncertainty comparison of the five-pointed star trajectory under three tracking strategies. (e) Localization results on the helical trajectory, where the magnet orientation changes periodically, resulting in distinct sensor trajectories for the two methods. (f) Zoomed-in view of the helical trajectory tracking results. (g) Position and orientation errors along the helical trajectory. (h) Uncertainty comparison of the helical trajectory under three tracking strategies.

errors are 1.98 mm/2.08°, 0.72 mm/1.55°, and 0.67 mm/1.50° for static, projection-based, and min-uncertainty tracking, respectively [see Fig. 18(c)]. For the helical trajectory, the errors are 1.59 mm/2.65°, 0.69 mm/1.62°, and 0.51 mm/1.00° [see Fig. 18(g)]. The associated uncertainty maps are shown in Fig. 18(d) and (h). Across both trajectories, the min-uncertainty method reduces position error by 67.04% and 16.51%, and orientation error by 43.87% and 20.73% compared to the static and projection-based methods, respectively. These results clearly demonstrate the effectiveness of minimizing localization uncertainty in enhancing the accuracy and robustness of magnetic localization.

#### D. Phantom Experiment

In this section, we present two phantom experiments: one focusing on interlocking nail alignment and the other on magnetic capsule localization. These experiments demonstrate the effectiveness of the proposed system and algorithm in enhancing localization accuracy, expanding the operational workspace, and showcasing their potential for medical applications.

1) *Locking Screw Alignment*: Distal screw alignment is a critical yet technically demanding step in interlocking nail surgery [37]. After the insertion of the interlocking nail into the medullary cavity of a long bone, distal locking screws must be accurately aligned with predesigned holes on the nail to ensure rotational and axial stability. However, the distal end of the nail is buried deep within the bone and lacks direct visual

or tactile access, making alignment a nontrivial task. Surgeons often rely on intraoperative fluoroscopy, repeated drilling, or mechanical targeting guides to achieve proper alignment, which are time-consuming, expose patients and staff to radiation, and carry risks of cortical bone damage due to misaligned drilling. To overcome these limitations, magnetic localization presents a promising, radiation-free alternative, which estimates the position and orientation of the hidden distal holes with high precision. This approach not only improves targeting accuracy, but also streamlines the surgical workflow and reduces radiation exposure.

To demonstrate the potential of the proposed mobile magnetic localization system and min-uncertainty localization in surgical scenarios, we apply it to the distal screw alignment task in interlocking nail surgery. Unlike most previous approaches that attach a small number of magnetic sensors to targeting tools, which offer only relative guidance without absolute pose information [38], [39], [40]. We investigate the use of a movable magnetic sensor array to estimate the absolute pose of a screw hole (simulated by a small embedded magnet) and to guide a robotic arm for automated and precise screw alignment. This validates that adaptive tracking can substantially improve localization quality, which is crucial for the success of such procedures.

As shown in Fig. 19(a)–(c), we embed a cylindrical magnet ( $R = 3\text{ mm}$ ,  $L = 3\text{ mm}$ ) into the distal end of a foam tibia bone to mimic the screw hole on an interlocking nail. A 3 mm square sticker is placed at the magnet's center to facilitate alignment accuracy evaluation. A 40 mm M4 titanium locking screw is

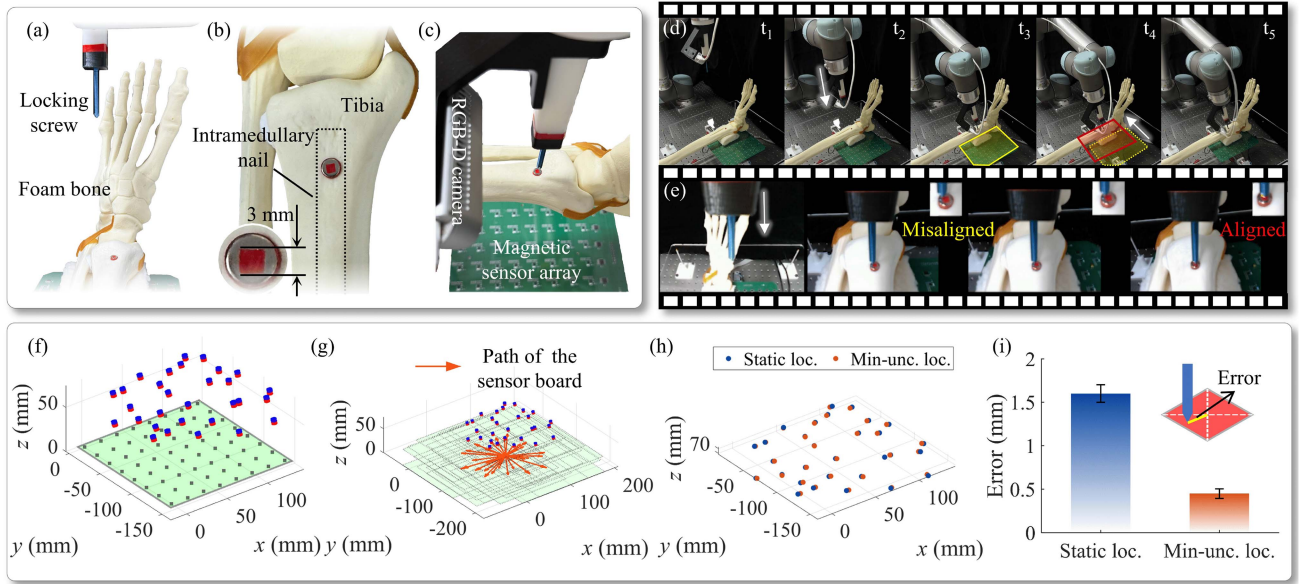


Fig. 19. Application of the proposed magnetic localization system in interlocking nail alignment. (a) Experimental setup including the foam bone and titanium screw mounted on the robotic arm. (b) Small magnet embedded in the distal tibia to simulate the screw hole; square sticker at its center serves for alignment evaluation. (c) Overall experimental setup. (d) One full trial involves: static localization ( $t_1$ ), initial screw insertion ( $t_2$ – $t_3$ ), sensor array movement for uncertainty minimization ( $t_4$ ), and refined screw alignment ( $t_5$ ). (e) First-person camera view shows improved alignment after sensor movement. (f) Initial magnet poses relative to the array across 30 trials. (g) Sensor displacement vectors to reach minimum uncertainty. (h) Comparison of alignment errors between static and min-uncertainty localization.

mounted on a 3-D-printed holder attached to the tool center point (TCP) of a robotic arm. An Intel RealSense D435i camera is also installed on the arm to observe the alignment from a first-person perspective. The foam bone is placed on an acrylic platform above the sensor array to simulate the distal locking scenario.

As illustrated in Fig. 19(d), the experiment proceeds as follows. First, the magnet-embedded bone is placed roughly over the sensor array, without intentionally aligning it to the center, emulating real surgical procedure ( $t_1$ ). Subsequently, an initial localization is performed with the static array, and the robotic arm attempts screw insertion based on this estimate ( $t_2$ – $t_3$ ). The array is then repositioned via the min-uncertainty strategy, moving from a suboptimal (yellow) region to an optimized (red) region ( $t_4$ ). Finally, a refined localization is performed at the new position, and the screw is microadjusted accordingly ( $t_5$ ). As evident from the first-person camera view in Fig. 19(e), the alignment significantly improves after sensor repositioning, enabling the screw to better match the center of the simulated hole, thereby facilitating accurate fixation. A demonstration of multiple trials is provided in the Supplementary Video.

We conduct 30 randomized trials, with initial magnet poses (relative to the sensor array) shown in Fig. 19(f), uniformly distributed over the sensor board to ensure fair comparison. The displacements of the array for uncertainty minimization are shown in Fig. 19(g). Localization results before and after sensor movement are compared in Fig. 19(h). The alignment error, which is measured as the distance between the screw tip and the sticker center in the camera frame is dropped from 1.60 to 0.45 mm on average, a reduction of 71.88%, as shown in Fig. 19(i). This substantial improvement highlights the

effectiveness of min-uncertainty sensor movement in enhancing localization accuracy, underscoring the practical potential of mobile magnetic sensor arrays for orthopedic surgical applications.

2) *Magnetic Capsule Tracking*: Capsule endoscopy is also a major application area for magnetic localization. Accurate localization of the capsule helps clinicians correlate the real-time video feed with specific anatomical locations within the gastrointestinal tract, and many studies have been conducted in this domain [41]. However, most existing studies rely on static sensor arrays with low spatial density, typically comparable in size to an operating table, which can lead to high localization uncertainty. In this experiment, we simulate the motion of a magnetic capsule inside a human stomach and use a mobile magnetic sensor array guided by a min-uncertainty strategy to dynamically track the target.

As shown in Fig. 20(a) and Supplementary Video, a stomach model approximately 25 cm in length is placed above the sensor array. A cylindrical magnet ( $R = 5$  mm,  $L = 5$  mm) is enclosed in a silicone shell mimicking the capsule's shape and placed inside the stomach model. A thin nylon wire is used to slowly pull the magnet downward, simulating its rolling motion through the stomach. The sensor array simultaneously tracks the capsule in real time. The results are shown in Fig. 20(b) and (c), with the magnet moving 17.71, 115.08, and 17.70 mm along the  $x$ ,  $y$ , and  $z$  axes, respectively. To enable continuous and accurate tracking, the sensor array moved 122.83 mm in the  $y$ -direction, significantly extending the effective tracking area compared to a static sensor array, while also reducing localization uncertainty.

Fig. 20(d) visualizes the capsule's trajectory in the  $yz$  plane, overlaid with uncertainty surface. A close-up view of the last tracking point is provided in Fig. 20(e), showing that the 95%

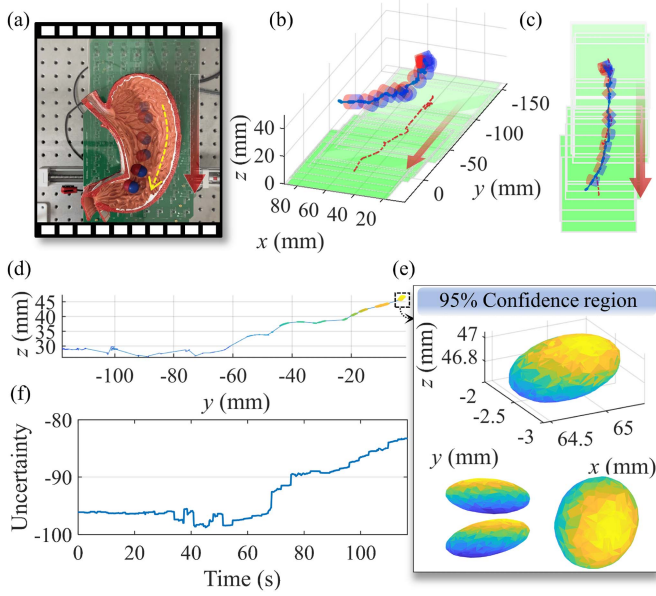


Fig. 20. Magnetic capsule localization experiment. (a) Overlay of physical motion paths of the capsule and sensor array. (b) Estimated capsule trajectory and the motion path of the sensor array. (c) Top-down view of the estimated trajectory. (d)  $yz$ -plane projection of the trajectory overlaid with uncertainty surface. (e) Zoomed-in view of the final trajectory point. (f) Localization uncertainty over time.

confidence region for the capsule's position remains within an acceptable range. The temporal evolution of localization uncertainty is plotted in Fig. 20(f), which correlates well with the varying distance between the capsule and the sensor array. This experiment demonstrates the potential of the proposed method for real-time localization of magnetic capsules in the gastrointestinal tract.

## V. CONCLUSION

This work presents a uncertainty analysis and mitigation of pose estimation on SE(3) for magnetic localization that explicitly accounts for sensor noise propagation. By modeling pose uncertainty in the Lie algebra and mapping it onto the manifold via the exponential map, we reveals the geometric structure of localization uncertainty, which is generally nonellipsoidal due to the coupling between position and orientation.

Base on this framework, we quantitatively analyze how configuration of the sensor array, target distance and orientation jointly shape localization uncertainty, and validate the analysis through simulations and experiments. These insights motivate the design of a planar, movable sensor array that actively minimizes localization uncertainty through real-time repositioning, thereby improving accuracy and extending the effective workspace.

The proposed magnetic localization system is demonstrated in path tracking experiments and representative medical scenarios, including distal locking screw alignment and magnetic capsule tracking, confirming its ability to achieve accurate and robust localization under constrained sensing resources. Together, these results highlight uncertainty-aware sensing and active sensor deployment as a general and effective paradigm for magnetic localization in robotic and clinical applications.

It is worth noting that, beyond sensor noise, magnetic localization is affected by additional parameters such as the magnetization magnitude. These parameters are treated as known and are therefore largely overlooked in most existing studies, yet they remain an important topic for further investigation. We believe that their impact can be quantified within the same local linearization framework adopted in this article, enabling a principled assessment of how parameter uncertainties propagate to the final localization estimates.

## APPENDIX A

### DERIVATION OF THE FIELD MODEL AND GRADIENT

In the magnetic field model of (2),  $P_1, P_2$  are the auxiliary functions, given in the following form:

$$P_1(k) = \mathcal{K} - \frac{2}{1-k^2}(\mathcal{K} - \mathcal{E}) \quad (20a)$$

$$P_2(k) = -\frac{\eta}{1-\eta^2}(\mathcal{P} - \mathcal{K}) - \frac{1}{1-\eta^2}(\eta^2\mathcal{P} - \mathcal{K}). \quad (20b)$$

$\mathcal{K}, \mathcal{E}$ , and  $\mathcal{P}$  in (20) are the complete elliptic integrals of the first, second, and third kinds defined as

$$\mathcal{K} = \mathcal{K}(\sqrt{1-k^2}) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-(1-k^2)\sin^2\theta}} \quad (21a)$$

$$\mathcal{E} = \mathcal{E}(\sqrt{1-k^2}) = \int_0^{\frac{\pi}{2}} \sqrt{1-(1-k^2)\sin^2\theta} d\theta \quad (21b)$$

$$\begin{aligned} \mathcal{P} &= \Pi(1-\eta^2, \sqrt{1-k^2}) \\ &= \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-(1-\eta^2)\sin^2\theta)\sqrt{1-(1-k^2)\sin^2\theta}}. \end{aligned} \quad (21c)$$

The other symbols in (2) and (20) are abbreviations as follows:

$$\zeta_{\pm} = z \pm L \quad (22a)$$

$$\alpha_{\pm} = [\zeta_{\pm}^2 + (\rho + R)^2]^{-\frac{1}{2}} \quad (22b)$$

$$\beta_{\pm} = \zeta_{\pm}\alpha_{\pm} \quad (22c)$$

$$\eta = \frac{\rho - R}{\rho + R} \quad (22d)$$

$$k_{\pm}^2 = \frac{\zeta_{\pm}^2 + (\rho - R)^2}{\zeta_{\pm}^2 + (\rho + R)^2}. \quad (22e)$$

Regarding the derivation of the magnetic gradient, we first express the five independent components of the magnetic gradient tensor in Cartesian coordinates using the partial derivatives in the cylindrical coordinate system, as follows:

$$\begin{cases} \partial_x B_x = \left(\frac{1}{\rho} - \frac{x^2}{\rho^3}\right) B_{\rho} + \frac{x^2}{\rho^2} \partial_{\rho} B_{\rho} \\ \partial_y B_y = \left(\frac{1}{\rho} - \frac{y^2}{\rho^3}\right) B_{\rho} + \frac{y^2}{\rho^2} \partial_{\rho} B_{\rho} \\ \partial_y B_x = -\frac{xy}{\rho^3} B_{\rho} + \frac{xy}{\rho^2} \partial_{\rho} B_{\rho} \\ \partial_z B_x = \frac{x}{\rho} \partial_z B_{\rho} \\ \partial_z B_y = \frac{y}{\rho} \partial_z B_{\rho} \end{cases} \quad (23)$$

Subsequently, by taking the partial derivative of (2a) and leveraging the differentiation properties of the complete elliptic



integrals, the expression in (23) is obtained

$$\begin{cases} \partial_\rho B_\rho = \frac{\mu_0 MR}{\pi} [\partial_\rho \alpha_+ P_1(k_+) + \alpha_+ A(k_+) \partial_\rho k_+^c \\ \quad - (\partial_\rho \alpha_- P_1(k_-) + \alpha_- A(k_-) \partial_\rho k_-^c)] \\ \partial_z B_\rho = \frac{\mu_0 MR}{\pi} [\partial_z \alpha_+ P_1(k_+) + \alpha_+ A(k_+) \partial_z k_+^c \\ \quad - (\partial_z \alpha_- P_1(k_-) + \alpha_- A(k_-) \partial_z k_-^c)] \end{cases} \quad (24)$$

The intermediate terms in (24) is as follows:

$$k_\pm^c = \sqrt{1 - k_\pm^2} \quad (25a)$$

$$\partial_\rho \alpha_\pm = - \frac{R + \rho}{(\zeta_\pm^2 + (R + \rho)^2)^{1.5}} \quad (25b)$$

$$\partial_\rho k_\pm^c = \sqrt{\frac{R}{\rho}} \frac{R^2 + \zeta_\pm^2 - \rho^2}{(\zeta_\pm^2 + (R + \rho)^2)^{1.5}} \quad (25c)$$

$$\partial_z \alpha_\pm = - \frac{\zeta_\pm}{(\zeta_\pm^2 + (R + \rho)^2)^{1.5}} \quad (25d)$$

$$\partial_z k_\pm^c = - \frac{2\zeta_\pm \sqrt{R\rho}}{(\zeta_\pm^2 + (R + \rho)^2)^{1.5}} \quad (25e)$$

$$\begin{aligned} A(k_\pm) = & \left( -\frac{1}{k_\pm^c} + \frac{4}{k_\pm^c{}^3} \right) K(k_\pm^c) \\ & + \frac{1}{k_\pm^c} \left( \frac{1}{k_\pm^c} - \frac{2}{k_\pm^c{}^3} \right) E(k_\pm^c) - \frac{2}{k_\pm^c{}^3} E(k_\pm^c). \end{aligned} \quad (25f)$$

For magnetic sources with more generalized geometries, the magnetic dipole model can be employed to characterize the surrounding magnetic field. Alternatively, this model can serve as a basis function to further enhance fitting precision in the near-field region. The magnetic field distribution and the corresponding gradient of the magnetic dipole model are expressed as follows:

$$\mathbf{B}_{\text{dipole}}(\mathbf{r}) = \frac{\mu_0}{4\pi\|\mathbf{r}\|} \left( 3\hat{\mathbf{r}}\hat{\mathbf{r}}^T - \mathbf{1}_{3\times 3} \right) \mathbf{m} \quad (26a)$$

$$\begin{aligned} \mathcal{H}_{\text{dipole}}(\mathbf{r}) = \nabla \mathbf{B}_{\text{dipole}}(\mathbf{r}) = & \frac{3\mu_0}{4\pi\|\mathbf{r}\|^7} \\ & (\|\mathbf{r}\|^2 (\mathbf{r}\mathbf{m}^T + \mathbf{m}\mathbf{r}^T + (\mathbf{m}^T \mathbf{r})\mathbf{1}_3) - 5\mathbf{m}^T \mathbf{r} (\mathbf{r}\mathbf{r}^T)) \end{aligned} \quad (26b)$$

where  $\mathbf{m}$  denotes the magnetic moment, the direction of which is coincident with the magnetization vector.

## APPENDIX B SENSOR FAULT DETECTION

To improve robustness against sporadic sensor faults in magnetic localization, we apply a  $\chi^2$  test on the measurement residuals [42]. The key idea is that, under the forward model and Gaussian measurement noise, the squared Mahalanobis distance follows a  $\chi^2$  distribution. Measurements that violate this distribution are treated as outliers and rejected.

We compute the squared Mahalanobis distance of sensor  $i$  at time  $k$  as

$$d_{i,k}^2 = \left( \tilde{\mathbf{B}}_{i,k} - \mathbf{B}_{i,k} \right)^T \mathbf{R}_s^{i-1} \left( \tilde{\mathbf{B}}_{i,k} - \mathbf{B}_{i,k} \right). \quad (27)$$

When the measurement model is consistent and the noise is Gaussian,  $d_{i,k}^2$  will approximately follow a  $\chi^2$  distribution with  $v$  degrees of freedom. For a triaxial magnetic sensor, we set  $v = 3$

$$d_{i,k}^2 \sim \chi^2(3). \quad (28)$$

Given a significance level  $\alpha$  we set the gating threshold  $\tau = \chi_{1-\alpha}^2(3)$ . For instance, with a common significance level of  $\alpha = 0.05$ , the gating threshold is  $\tau = \chi_{0.95}^2(3) \approx 7.81$ . We can therefore declare sensor  $i$  as abnormal at time  $k$  if

$$d_{i,k}^2 > \tau. \quad (29)$$

Flagged measurements are then excluded from the subsequent state update, or alternatively handled by covariance inflation. This  $\chi^2$  gating procedure provides a lightweight statistical safeguard that can be applied per sensor and per time step, enabling identification and suppression of faulty sensor readings during magnetic localization.

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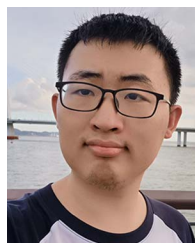
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